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More About General Saddlepoint Approximations

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In Easton and Ronchetti (1986), a method of generalized saddlepoint approximations is proposed and shown useful, especially in the case of small sample sizes. A possible improvment of the method is suggested to prevent its potential deficiences and increase its applicability. Easton and Ronchetti's approximation and its modified version are extended to bootstrap applications. These results provide a satisfactory answer to Davison and Hinkley's (1988) open question on the bootstrap distribution in the AR(1) model. Numerical examples show the great accuracy of the modified method even when the original approximation fails dramatically.

Autoregressive model, Beta distribution, Bootstrap, Edgeworth expansion, Mean estimation, Nonlinear statistics

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More about general saddlepoint approximations

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SUMMARY

In Easton and Ronchetti (1986), a method of general saddlepoint approximations is proposed and shown useful, especially in the case of small sample sizes. A possible improvement of the method is suggested to prevent its potential deficiences and increase its applicability. Easton and Ronchetti's approximation and its modified version are extended to bootstrap applications. These results provide a satisfactory answer to Davison and Hinkley's (1988) open question on the bootstrap distribution in the AR(1) model. Numerical examples show the great accuracy of the modified method even when the original approximation fails dramatically.

Key Words: Autoregressive model; Beta distribution; Bootstrap; Edgeworth expansion; Mean estimation; Nonlinear statistics.

1. INTRODUCTION

The technique of saddlepoint expansions, introduced into the statistics literature by Daniels (1954), has been shown to be an important tool in statistics. Among other papers, Barndorff-Nielsen and Cox (1979) and Reid (1988) provide excellent discussions on saddlepoint approximations in the parametric context. Applications to nonparametric analysis can be found in Robinson (1982) and Davison and Hinkley (1988), the latter applied the saddlepoint method to bootstrap and randomization problems. Most of the applications are limited to simple statistics, such as the sample mean, with the known cumulant generating functions (CGF), since the CGF's are explicitly needed in the calculations of the approximations. Recent extensions to some specific nonlinear statistics by Srivastava and Yau (1989) and Wang (1990b) require similar conditions.

An alternative approach was proposed by Easton and Ronchetti (1986), who use the first four terms of the Taylor series expansion to approximate the CGF. Thus, only the first four cumulants are required for saddlepoint approximations. We now briefly review this approach.

Suppose that interest is in the density $f_n(x)$ and the cumulative distribution $F_n(x)$ of some statistic $V_n(X_1, \ldots, X_n)$, where $X_1, \ldots, X_n$ are iid observations. Assume that the cumulant generating function $K_n(t)$, which may be unknown, of V_n exists for real t in some nonvanishing

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interval that contains the origin. Let κ_{in} be the ith cumulants of V_n, i = 1, 2, 3, 4. Then $\kappa_{1n} = \mu_n = E(V_n)$ and $\kappa_{2n} = \sigma_n^2 = \text{var}(V_n)$. When $K_n(t)$ is unknown or difficult to evaluate, Easton and Ronchetti (1986) propose to use

$$\tilde{K}_n(t) = \kappa_{1n}t + \frac{\kappa_{2n}}{2!}t^2 + \frac{\kappa_{3n}}{3!}t^3 + \frac{\kappa_{4n}}{4!}t^4 \tag{1}$$

to approximate $K_n(t)$. The resulting saddlepoint apporoximations are, for $f_n(x)$,

$$\tilde{f}_n(x) = \left[\frac{n}{2\pi\,\tilde{R}''(T_0)}\right] e^{n\{\tilde{R}_n(T_0) - T_0x\}} \tag{2}$$

and, for $F_n(x)$,

$$\tilde{F}_n(x) = \Phi(\tilde{w}) + \phi(\tilde{w})\left(\tilde{w}^{-1} - \tilde{z}^{-1}\right), \tag{3}$$

where $\tilde{R}_n(T) = \tilde{K}_n(nT)/n$, T_0 is determined as a solution to

$$\tilde{R}_n'(T_0) = x, \tag{4}$$

ϕ and Φ are the standard normal density and distribution function respectively, $\tilde{w} = [2n\{T_0x - \tilde{R}_n(T_0)\}]^{1/2}\text{sgn}(T_0)$ and $\tilde{z} = T_0\{n\tilde{R}_n''(T_0)\}^{1/2}$. Formulas (2) and (3) correspond to the classical saddlepoint formulas; see Daniels (1987). Formula (3) was not explicitly given in Easton and Ronchetti (1986). Instead they used numerical integrations over (2) to approximate F_n. Under mild regularity conditions on V_n to ensure the validity of the Edgeworth expansion, using the Edgeworth and saddlepoint expansions (Easton and Ronchetti, 1986), it is easily shown that approximations (2) and (3) have relative error of $O(n^{-1})$ for all x such that $|x - \mu_n| \le d/n^{1/2}$ for any fixed constant d. Numerical examples (Easton and Ronchetti, 1986; Wang, 1990b) show that these approximations are often satisfactory and they overcome some deficiencies that the Edgeworth approximations could have, such as negative tail probabilities.

A drawback of the approximations is that, contrary to Easton and Ronchetti's claim, the solution to (4) may not be unique. This phenomenon could happen more often in bootstrapping applications where the cumulants are estimated. In such cases, the approximations could fail to work in tail areas in which we are particularly interested.

In this note, we first extend Section 2. To a greater extent, Section 3 considers a simple modification of their method, which avoids the problem of multiple roots and at the same time retains the same order of the accuracy. The modified method is then extended to bootstrap applications in Section 4. A satisfactory answer to the open question on the bootstrap distribution in the AR(1) model raised by Davison and Hinkley (1988) is obtained by straightforward applications of the results. Some numerical examples are given.

2. BOOTSTRAP APPLICATIONS

Our goal is to approximate

$$F_n(x) = \mathrm{pr}(V_n \le x \mid F)\ , \tag{5}$$

where $V_n = V_n(X_1, \ldots, X_n)$ is a (approximate) pivotal quantity (Hinkley, 1988), F is the underlying distribution of the observations; e.g., in the location problem let $V_n = \overline{X} - \mu$. Let $\hat{F}$ be the empirical distribution function. A bootstrap method is to approximate $F_n(x)$ by the bootstrap distribution

$$\hat{F}_n(x) = \mathrm{pr}(V_n^* \le x \mid \hat{F})\ , \tag{6}$$

where $V_n^* = V_n(X_1^*, \ldots, X_n^*)$, X_i^* are sampled from $\hat{F}$ (e.g., in the location problem, $V_n^* = \overline{X}^* - \overline{X}$). The exact values for $\hat{F}_n(x)$ are usually difficult to obtain and therefore approximations are desirable. Numerical simulation is a simple method but it could be costly. In the location problem and other simple situations, Davison and Hinkley (1988) suggest a very efficient alternative, namely, bootstrap saddlepoint approximations. The main step is to use the conditional (given the data) CGF,

$$\hat{K}(t) = \log\{\tfrac{1}{n} \sum_1^n \exp(tX_i)\}\ , \tag{7}$$

to replace K(t) in the classical saddlepoint formulas (Daniels, 1987).

For more general V_n, it is possible to extend Easton and Ronchetti's method similarly as they have suggested as a further research problem. Let $\hat{\kappa}_{in}$ be the ith conditional cumulants of V_n^* given $\hat{F}$, $i = 1, 2, 3, 4$, and

$$\tilde{\hat{K}}(t) = \hat{\kappa}_{1n} t + \frac{\hat{\kappa}_{2n}}{2!} t^2 + \frac{\hat{\kappa}_{3n}}{3!} t^3 + \frac{\hat{\kappa}_{4n}}{4!} t^4\ . \tag{8}$$

Replacing $\tilde{K}_n$ by $\tilde{\hat{K}}_n$ in (3), we have the corresponding saddlepoint formula for $\hat{F}_n(x)$:

$$\tilde{\hat{F}}_n(x) = \Phi\left(\tilde{\hat{w}}\right) + \phi\left(\tilde{\hat{w}}\right)\left(\tilde{\hat{w}}^{-1} - \tilde{\hat{z}}^{-1}\right), \tag{9}$$

where $\tilde{\hat{w}} = \left[2n\{T_0x - \tilde{\hat{R}}_n(T_0)\}\right]^{1/2} \mathrm{sgn}(T_0)$, $\tilde{\hat{z}} = T_0\{n\, \tilde{\hat{R}}_n''(T_0)\}^{1/2}$, $\tilde{\hat{R}}_n(T) = \tilde{\hat{K}}_n(nT)/n$ and T_0 is a solution to

$$\tilde{\tilde{R}}'_n(T_0) = x. \tag{10}$$

Using an argument of the Edgeworth and saddlepoint expansions parallel to that in Section 2 of Easton and Ronchetti (1986) and by a treatment on the discreteness problem similar to that in Wang (1990a), it is easily seen that with a negligible numerical error, uniformly

$$\tilde{\hat{F}}_n(x) = \hat{F}_n(x)\{1 + O_p(n^{-1})\} , \tag{11}$$

for $| x - \hat{\kappa}_{1n} | \leq d/n^{1/2}$, i.e., the error term is of order n^{-1} uniformly, aside from a very small numerical error caused by the discreteness originated from $\hat{F}$. The following example illustrates the usefulness of the approximation.

Example 1. Assume that data are modelled by the autoregressive process

$$X_i = \theta X_{i-1} + \epsilon_i \ \ (i = 1, \ldots, n), X_0 \equiv 0 ,$$

where ϵ_i have a common symmetric distribution, and that one wishes to test H_0: θ=0. A test statistic is

$$W_n = \sum_2^n X_{i-1}X_i \ / \sum_1^n X_i^2 .$$

Notice that under H_0, $E(W_n) = 0$ and W_n is independent of the scale parameter. The corresponding resampled statistic is

$$W_n^* = \sum_2^n X_{i-1}^* \, X_i^* \ / \sum_1^n X_i^{*2},$$

where X_i^* are independently resampled from ($\pm X_1, \ldots, \pm X_n$) due to its symmetry. Davison and Hinkley (1988) have considered this problem by using the simpler resampling scheme of randomization and raised the open question about the distribution of W_n^* under the above bootstrap resampling scheme. Noticing that

$$\hat{F}_n(x) = \mathrm{pr}\Big(W_n^* \leq x \mid \hat{F}\Big) = \mathrm{pr}\Big(V_n^* \leq z \mid \hat{F}\Big) , \tag{12}$$

where $V_n^* = \Big(\Sigma X_{i-1}^* X_i^* - \Sigma Y_i^*\Big)/n$, $Y_i^* = x\Big(X_i^{*2} - \Sigma X_i^2/n\Big)$ and $z = x\Sigma X_i^2/n$, we need only to focus on V_n^*. It is easily obtained that

$$\hat{\kappa}_{1n} = 0 ,$$

$$\hat\kappa_{2n} = \frac{n-1}{n^4}\left(\Sigma X_i^2\right)^2 + \frac{1}{n^2}\Sigma Y_i^2 ,$$

$$\hat\kappa_{3n} = -\frac{6(n-1)}{n^5}\left(\Sigma X_i^2\right)\left(\Sigma X_i^2 Y_i\right) - \frac{1}{n^3}\Sigma Y_i^3 ,$$

$$\hat\kappa_{4n} = \frac{n-1}{n^6}\left(\Sigma X_i^4\right)^2 + \frac{6(n-2)}{n^7}\left(\Sigma X_i^4\right)\left(\Sigma X_i^2\right)^2 - \frac{3(3n-5)}{n^8}\left(\Sigma X_i^2\right)^4$$

$$+ \frac{12(n-1)}{n^6}\left(\Sigma X_i^2 Y_i\right)^2 + \frac{1}{n^4}\Sigma Y_i^4 - \frac{3}{n^5}\left(\Sigma Y_i^2\right)^2 ,$$

where $Y_i = x(X_i^2 - \Sigma X_i^2/n)$. It is now straightforward to apply formula (9). To illustrate, consider the following data set (n=33) from Ogbonmwan and Wynn (1988), which was simulated from Normal (0,1) errors under H_0:

(0)	−0.625	−0.631	0.290	−1.402	−0.684	0.562	2.737	−0.027
−0.085	−0.151	−0.766	0.415	0.490	1.222	−1.590	−0.262	−2.001
0.679	−1.128	1.075	−0.206	−1.447	−2.287	−1.468	0.0415	1.166
−1.270	−1.712	−1.391	−0.263	1.386	−0.278	−1.343		

The "exact" bootstrap distribution, obtained by 10^5 simulations, and the saddlepoint and normal approximations are given in Table 1. The modified saddlepoint approximation will be addressed in the next section. The example shows that the saddlepoint approximations are very satisfactory except in the extreme area where it gets slightly worse, but is still much better than the normal approximation.

3. MODIFIED METHOD

It is possible that (4) or (10) have multiple roots for various x. A simple example is the sample mean of the Bernoulli distribution. When the parameter $p=\frac{1}{2}$ we have $\kappa_{1n} = \frac{1}{2}$, $\kappa_{2n} = \frac{1}{4n}$, $\kappa_{3n} = 0$, $\kappa_{4n} = \frac{1}{8n^3}$ and therefore

$$\tilde R_n'(T) = \tfrac{1}{2} + \frac{T}{4} - \frac{T^3}{8\cdot 3!} .$$

It is clear that multiple roots to (4) exist. More examples will be discussed shortly. In such problematical case, it is natural to select the root nearest to zero. However, when x moves away from the mean, a proper solution may not exist. This phenomenon may cause a considerable problem as we will see in the examples.

We now propose a simple modification to prevent such undesirable feature while retaining the validity of the numerical accuracy as well as the asymptotic property. Let

$$\tilde{K}_n(t;\, b) = \kappa_{1n} t + \frac{\kappa_{2n}}{2!}\, t^2 + \left(\frac{\kappa_{3n}}{3!}\, t^3 + \frac{\kappa_{4n}}{4!}\, t^4\right) g_b(t)\,, \tag{13}$$

where $g_b(t) = \exp\{-t^2/(n^3 \kappa_{2n} b^2)\}$ and b is a properly chosen constant. The modified method is to replace $\tilde{K}_n(t)$ in (1) by $\tilde{K}_n(t;\, b)$, obtaining modified $\tilde{f}_n(x;\, b)$ and $\tilde{F}_n(x;\, b)$ corresponding to (2) and (3), respectively. Notice that the solution $T = T_0^{(b)}$ to

$$\tilde{R}_n'(T,\, b) = x \tag{14}$$

is always unique for a proper domain of x and suitable b, where $\tilde{R}_n(T;\, b) = \tilde{K}_n(nT; b)/n$. By suitable rescaling we assume that $\kappa_{in} = O(n^{1-i})$, i = 2, 3, 4. When $t = O(n^{1/2})$, $g_b(t) = 1 + O(n^{-1})$ and therefore $\tilde{K}_n(t;\, b) = \tilde{K}_n(t) + O(n^{-3/2})$. Thus it is easily shown as in Section 2 of Easton and Ronchetti (1986) that when $|\, x - \mu_n| \le d/n^{1/2}$, $nT_0^{(b)} = O(n^{1/2})$ and therefore $\tilde{f}_n(x;\, b)$ and $\tilde{F}_n(x;\, b)$ are correct up to $O(n^{-1})$ uniformly.

Note that asymptotically b can be any fixed constant in (13). But in practice we suggest that b be $2^{3/2}$ or a suitably larger constant so that $\tilde{R}_n'(T;\, b)$ is strictly increasing (i.e., $\tilde{R}_n''(T;\, b) > 0$) in a sufficiently large interval U on the x-axis on which the distribution function is desired. If in fact $\tilde{R}_n''(T) > 0$ on such U and thus $b = 2^{3/2}$, the modification has little effect. This phenomenon is supported by the calculations of $\tilde{F}_n(x;\, b)$ in the two examples of L statistics in Easton and Ronchetti (1986) as well as other examples. Notice that $\tilde{F}_n(x;\, b) \to \tilde{F}_n(x)$ as $b \to 0$ whereas it approaches to the normal approximation as $b \to \infty$. It is therefore always possible to find b large enough to guarantee $\tilde{R}_n''(T;\, b) > 0$ on U.

We now consider an example where the modification is shown to be useful. In the illustration we focus on the cumulative distribution.

Example 2. In this example we wish to approximate the distribution of sample mean of the Beta distribution. Let $X_1, \ldots, X_n$ be a sample from B(p,q). The moment generating function is

$$M(t) = \frac{\Gamma(p+q)}{\Gamma(p)} \sum_{j=0}^{\infty} \frac{\Gamma(p+j)\, t^j}{\Gamma(p+q+j)\, \Gamma(j+1)}\,,$$

which makes it very difficult to calculate the classical saddlepoint approximations. It is, however, easy to obtain the central moments of $\bar{X}$ (Johnson and Kotz, 1970, p.40):

$$\mu'_{in} = \frac{p^{[i]}}{(p+q)^{[i]} n^{i-1}} , \qquad i = 1, 2, 3, 4,$$

where $y^{[i]} = y(y+1) \ldots (y+i-1)$ is the ascending factorial.

To be specific, let p = 2, q = 5, and n = 5. Then simple calculations show that $\kappa_{1n} = 2/7$, $\kappa_{2n} = 0.005102$, $\kappa_{3n} = 9.718 \times 10^{-5}$, $\kappa_{4n} = -6.247 \times 10^{-7}$. Table 2 provides the original and the modified ($b = 2^{3/2}$) Easton and Ronchetti approximations, the normal and the second order Edgeworth approximations and the "exact" values obtained by 10^5 simulations. This table shows that Easton and Ronchetti's method works well for $x \geq 0.18$ but starts losing its accuracy at $x = 0.17$. For $x < 0.17$, no suitable solution T_0 to (4) exists, so that the method fails to work at all. The modified method overcomes this problem and continues to provide very accurate apporximation for $x \leq 0.17$. It has the bes' overall performance among the listed methods.

4. MODIFIED METHOD IN BOOTSTRAP APPLICATIONS

It is straightforward to extend the modified method to the bootstrap context. Referring to (13), let

$$\tilde{\hat{K}}_n(t; b) = \hat{\kappa}_{1n} t + \frac{\hat{\kappa}_{2n}}{2!} t^2 + \left(\frac{\hat{\kappa}_{3n}}{3!} t^3 + \frac{\hat{\kappa}_{4n}}{4!} t^4 \right) \hat{g}_b(t) , \tag{15}$$

where $\hat{g}_b(t) = \exp\{-t^2/(n^3 \hat{\kappa}_{2n} b^2)\}$ and the constant b is chosen according to the same suggestions as in Section 3. Replacing $\hat{K}(t)$ by $\tilde{\hat{K}}(t; b)$ in (9) we obtain the modified saddlepoint formula $\tilde{\hat{F}}_n(x; b)$ for the bootstrap distribution $\hat{F}_n(x)$. Combining the results in Sections 2 and 3, we obtain that for $x - \mu_n = O(n^{-1/2})$

$$\tilde{\hat{F}}_n(x; b) = \hat{F}_n(x) \{1 + O_p(n^{-1})\} , \tag{16}$$

aside from a negligibly small numerical error due to discreteness. As in the parametric case, the modification has little effect when the original method works well; see Table 1. However, it may provide substantial improvement otherwise.

Example 3. We continue to consider the problem in Example 1, but assume that n = 10. The following data set was simulated from the mixture of normals $\frac{1}{2}N(-1, 0.2) + \frac{1}{2}(1, 0.2)$:

−1.240	−0.912	1.098	−1.209	0.838	−1.115	1.088	0.806	−0.896

The same calculations were performed as in Example 1. The results given in Table 3 show a pattern of the original and the modified saddlepoint methods that is similar to the one in Table 2. This example demonstrates the applicability of the modified method in nonlinear bootstrap problems.

Example 4. In the final example, we make a comparison between the modified method and Davison and Hinkley's (1988) boostrap saddlepoint method in the setting of bootstrapping a sample mean. Let

$$W_n^* = \overline{X}^* - \overline{X}$$

corresponding to $W_n = \overline{X} - \mu$. Given the sample of n = 10 values (Davison and Hinkley, 1988),

9.6 10.4 13.0 15.0 16.6 17.2 17.3 21.8 24.0 33.8

it is easily calculated that $\hat{\kappa}_{1n} = 0$, $\hat{\kappa}_{2n} = 4.6532$, $\hat{\kappa}_{3n} = 3.2094$ and $\hat{\kappa}_{4n} = 0.95147$. Table 4 is reproduced from Table 1 of Davison and Hinkley (1988) except for the entries for Easton and Ronchetti's approximation (ERS) and its modified version (MERS) with $b = 2^{3/2}$. The modified approximation is almost as good as that of Davison and Hinkley (DHS), although it uses relatively less information from the data and requires relatively less computing time. Notice that MERS is slightly better (worse) than DHS in the upper (lower) extreme tail and it is slightly wider in the extreme tails. Easton and Ronchetti's method works as well as MERS on the upper tail until x gets close to -3.5 where its performance gets worse sharply. Note that $\text{pr}(\overline{X}^* - \overline{X} \le -3.5 \mid \hat{F}) \doteq 0.04$. No suitable solution to (10) exists for $x \le -4.01$. The normal and Edgeworth approximations which are given in Davison and Hinkley (1988) are not as accurate as those of DHS and MERS.

5. CONCLUDING REMARKS

In this note we have developed extensions of Easton and Ronchetti's (1986) useful method for approximating the distributions of general statistics. In some cases, Easton and Ronchetti's method fails to work since the approximation (1) or (8) of the CGF may not satisfy the condition of convexity which is essential in the saddlepoint technique. The theoretically justified modifications remedy this deficiency and widen the applicability, as supported by the numerical examples. Using the new developments we have been able to provide a satisfactory answer to Davison and Hinkley's (1988) open question on the bootstrap distribution of the test statistic in the AR(1) model. Moreover, it is suggested by Example 2 that in the case of sample mean, Easton and Ronchetti's method or its modifications could give satisfactory results while the classical formulas are difficult to calculate.

The methcds are easily implemented in a short FORTRAN program. The roots required in the approximations can be found by a procedure of bisection search. Example 4 shows that in some simple bootstrap problems, the modified method accomplishes nearly as much as Davison and Hinkley's method but requires less calculations. Nevertheless, we reemphasize Easton and Ronchetti's advice that whenever the CGF is known and easy to implement, one should use the classical formulas.

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Table 1. Approximations to the distribution in the AR(1) model; n = 33.

x	Exact* $\hat{F}_n$	Saddlepoint	Modified Saddlepoint	Normal
−.50	.0010	.0029	.0029	.0072
−.45	.0028	.0053	.0054	.0117
−.40	.0076	.0100	.0101	.0192
−.35	.0173	.0191	.0192	.0316
−.30	.0364	.0365	.0366	.0515
−.25	.0687	.0672	.0673	.0827
−.20	.1192	.1165	.1163	.1294
−.15	.1893	.1871	.1871	.1952
−.10	.2790	.2781	.2780	.2815
−.05	.3849	.3848	.3848	.3855
−.02	.4536	.4534	.4534	.4536

* obtained by 10^5 simulations

Table 2. Approximations to $pr(\overline{X} \leq x)$ for Beta (2, 5); n = 5.

x	Exact* F_n	Saddlepoint	Modified Saddlepoint	Normal	Edgeworth
.07	.0001	___	.0002	.0013	−.0001
.09	.0006	___	.0007	.0031	.0003
.12	.0039	___	.0039	.0102	.0043
.15	.0189	___	.0177	.0287	.0202
.16	.0291	___	.0275	.0392	.0305
.17	.0430	.0534	.0411	.0526	.0444
.18	.0608	.0672	.0592	.0694	.0623
.20	.1118	.1155	.1103	.1151	.1124
.25	.3222	.3238	.3229	.3085	.3231
.30	.5945	.5947	.5946	.5793	.5948
.40	.9375	.9382	.9378	.9452	.9383
.50	.9971	.9971	.9972	.9987	.9973
.55	.9996	.9996	.9996	.9999	.9997
.58	.9999	.9999	.9999	1.0000	1.0001

* obtained by 10^5 simulations; "___" not available.

Table 3. Approximations to the distribution in the AR(1) model; n = 10.

x	Exact* $\hat{F}_n$	Saddlepoint	Modified Saddlepoint	Normal
−.90	.0003	___	.0001	.0019
−.80	.0019	___	.0008	.0048
−.75	.0029	___	.0030	.0074
−.70	.0083	___	.0088	.0112
−.65	.0163	___	.0141	.0168
−.60	.0203	___	.0219	.0246
−.59	.0210	___	.0239	.0265
−.58	.0220	.0346	.0259	.0285
−.50	.0488	.0503	.0484	.0497
−.40	.0918	.0949	.0940	.0928
−.30	.1653	.1638	.1634	.1597
−.20	.2565	.2578	.2576	.2529
−.10	.3732	.3728	.3728	.3695

* obtained by 10^5 simulations; "___" not available

Table 4. Approximations to resampling percentage points of $\overline{X} - \mu$

Probability	Exact*	DHS	ERS	MERS	Normal
.0001	−6.34	−6.31	___	−6.56	−8.46
.0005	−5.79	−5.78	___	−5.88	−7.48
.001	−5.55	−5.52	___	−5.56	−7.03
.005	−4.81	−4.81	___	−4.77	−5.86
.01	−4.42	−4.43	___	−4.39	−5.29
.05	−3.34	−3.33	−3.38	−3.31	−3.74
.10	−2.69	−2.69	−2.71	−2.68	−2.91
.20	−1.86	−1.86	−1.87	−1.86	−1.91
.80	1.80	1.80	1.79	1.79	1.91
.90	2.87	2.85	2.84	2.85	2.91
.95	3.73	3.75	3.74	3.74	3.74
.99	5.47	5.48	5.48	5.47	5.29
.995	6.12	6.12	6.13	6.12	5.86
.999	7.52	7.46	7.51	7.48	7.03
.9995	8.19	7.99	8.06	8.02	7.48
.9999	9.33	9.12	9.26	9.18	8.46

* obtained by $5X10^4$ simulations; "___" not available